PERTH MODERN SCHOOL
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Independent Public School

Course Specialist Year 11

Student name: ____________________________ Teacher name: _________________

Date: 22 July 2022

Task type: **Response**

Time allowed for this task: **_____40______ mins**

Number of questions: **_____6______**

Materials required: **Calculator-Free**

Standard items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: Drawing instruments, templates

Marks available: **__40____ marks**

Task weighting: **__10__%**

Formula sheet provided: **Yes**

Note: All part questions worth more than 2 marks require working to obtain full marks.

Question 1 (2.2.1, 2.2.2) (6 marks)

If $A = \begin{bmatrix} 3 & -1 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 2 \\ 5 & 2 \end{bmatrix}$

(a) Determine the matrix X such that $2A - X = B$ (3 marks)

Solution
$X = 2A - B$ $= 2\begin{bmatrix} 3 & -1 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} -3 & 2 \\ 5 & 2 \end{bmatrix}$ $= \begin{bmatrix} 6 & -2 \\ 2 & 8 \end{bmatrix} - \begin{bmatrix} -3 & 2 \\ 5 & 2 \end{bmatrix}$ $= \begin{bmatrix} 9 & -4 \\ -3 & 6 \end{bmatrix}$
Specific behaviours
✓ correct expression for X in terms of A and B   ✓ correct elements of 2A   ✓ correct elements of X

(b) Determine AB (3 marks)

Solution
$AB = \begin{bmatrix} 3 & -1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ 5 & 2 \end{bmatrix}$ $= \begin{bmatrix} (3)(-3) + (-1)(5) & (3)(2) + (-1)(2) \\ (1)(-3) + (4)(5) & (1)(2) + (4)(2) \end{bmatrix}$ $= \begin{bmatrix} -14 & 4 \\ 17 & 10 \end{bmatrix}$
Specific behaviours
✓ one correct element of AB   ✓ two correct elements of AB   ✓ all correct elements of AB

Question 2 (1.3.3) (5 marks)

Let n be an integer. Prove that $n+2$ is even if and only if $n+1$ is odd.

Solution
($\Rightarrow$): If $n+2$ is even, then $n+2=2k$, where $k \in Z$. $n+1 = 2k-1$ $= 2k-2+1$ $= 2(k-1)+1$ Hence, $n+1$ is odd. ($\Leftarrow$): If $n+1$ is odd, then $n+1=2k+1$, where $k \in Z$. $n+2 = 2k+2$ $= 2(k+1)$ Hence, $n+2$ is even.
Specific behaviours
✓ recognises the need to prove both $\Rightarrow$ and $\Leftarrow$ ✓ writes $n+2$ as $2k,\ k \in Z$ ✓ proves that $n+1$ is odd in terms of $2Z+1$ ✓ writes $n+1$ as $2k+1,\ k \in Z$ ✓ proves that $n+2$ is even in terms of $2Z$

Question 3 (1.3.4, 1.3.5) (9 marks)

Write whether each of the following statement is true or false, then prove or disprove it accordingly.

(a) $\forall\, n \in N,\ n^2 - n + 7$ is prime. (3 marks)

Solution
The statement is **false**, and is disproved with the following counterexample: Let $n = 7$. Then $n^2 - n + 7 = 49 = 7 \times 7$, which is not prime.
Specific behaviours
✓ states false ✓ states counterexample with a particular value of n ✓ shows that for that value of n, $n^2 - n + 7$ is not prime.

(b) For all irrational numbers p and q, where $p \neq q$, $\frac{p}{q}$ is always irrational.

(3 marks)

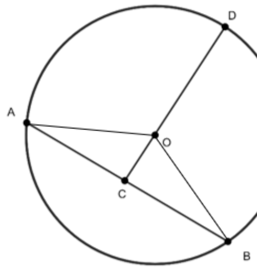
Solution
The statement is **false**, and is disproved with the following counterexample: Let $p = \sqrt{8}$ and $q = \sqrt{2}$. Then, $\frac{p}{q} = \frac{\sqrt{8}}{\sqrt{2}} = \frac{\sqrt{4}\times\sqrt{2}}{\sqrt{2}} = 2$ which is rational.
Specific behaviours
✓ states false ✓ states counterexample with particular values of p and q ✓ shows that for those values of p and q, $\frac{p}{q}$ is not irrational.

(c) There exist two different irrational numbers such that their sum is rational. (3 marks)

Solution
The statement is **true**, and is proved with the following example: Let the two irrational numbers be $\sqrt{2}$ and $\left(1 - \sqrt{2}\right)$. Then $\sqrt{2} + \left(1 - \sqrt{2}\right) = 1$ which is rational.
Specific behaviours
✓ states true ✓ states example with particular values of two different irrational numbers and ✓ shows that for those two different irrational numbers, the sum is rational.

Question 4 (1.3.15) (6 marks)

A circle with centre O is shown below (not drawn to scale). Given its radius is r and chord $AB = x$ with $AB \perp CD$, and centre O is on CD



(a) prove that CD bisects AB. (2 marks)
[Hint: Use congruent triangles]

Solution
(Connect OA and OB) For ΔAOC and ΔBOC: $\angle OCA = \angle OCB = 90°$ (given) $OA = OB$ (radii) OC is common $\therefore \Delta AOC \equiv \Delta BOC$ (RHS) $\therefore AC = BC$ Hence, CD bisects AB
Specific behaviours
✓ prove $\Delta AOC \equiv \Delta BOC$ with correct congruence tests ✓ states $AC = BC$

(b) show that $CD = \frac{2r+\sqrt{4r^2-x^2}}{2}$. (4 marks)

Solution
$OA = r$ and $AC = \frac{x}{2}$ $OC = \sqrt{r^2 - \left(\frac{x}{2}\right)^2}$ (Pythagoras' theorem) $= \sqrt{r^2 - \frac{x^2}{4}} = \sqrt{\frac{4r^2 - x^2}{4}} = \frac{\sqrt{4r^2 - x^2}}{2}$ $CD = r + \frac{\sqrt{4r^2 - x^2}}{2}$ $= \frac{2r + \sqrt{4r^2 - x^2}}{2}$
Specific behaviours
✓ states $AC = \frac{x}{2}$ ✓ expresses OC in terms of r and x using Pythagoras' ✓ simplifies expression for OC ✓ expresses CD as r plus OC and simplifies

Question 5 (2.3.4, 2.3.5) (7 marks)

Use mathematical induction to prove that $\frac{1}{2\times5}+\frac{1}{5\times8}+\cdots+\frac{1}{(3n-1)(3n+2)}=\frac{n}{6n+4}$ for all $n \in Z^+$.

Solution
Let $P(n)$ denote the proposition '$\frac{1}{2\times5}+\frac{1}{5\times8}+\cdots+\frac{1}{(3n-1)(3n+2)}=\frac{n}{6n+4}$' for all $n \in Z^+$. For $P(1)$: LHS $=\frac{1}{2\times5}=\frac{1}{10}$ RHS $=\frac{1}{6\times1+4}=\frac{1}{10}$ LHS=RHS, so $P(1)$ is true. Assume that $P(k)$ is true for some positive integer k. Then $\frac{1}{2\times5}+\frac{1}{5\times8}+\cdots+\frac{1}{(3k-1)(3k+2)}=\frac{k}{6k+4}$. Now LHS of $P(k+1)=\frac{1}{2\times5}+\frac{1}{5\times8}+\cdots+\frac{1}{(3k-1)(3k+2)}+\frac{1}{(3(k+1)-1)(3(k+1)+2)}$ $=\frac{k}{6k+4}+\frac{1}{(3k+2)(3k+5)}$ $=\frac{k}{2(3k+2)}+\frac{1}{(3k+2)(3k+5)}$ $=\frac{k}{2(3k+2)}+\frac{1}{(3k+2)(3k+5)}$ $=\frac{k}{2(3k+2)}\times\frac{(3k+5)}{(3k+5)}+\frac{1}{(3k+2)(3k+5)}\times\frac{2}{2}$ $=\frac{3k^2+5k+2}{2(3k+2)(3k+5)}$ $=\frac{(3k+2)(k+1)}{2(3k+2)(3k+5)}$ $=\frac{k+1}{6k+10}$ $=\frac{k+1}{6(k+1)+4}$ $=$ RHS of $P(k+1)$ $\therefore P(k+1)$ is true. Hence, by the principle of mathematical induction, $P(n)$ is true for all positive integers n.
Specific behaviours
✓ proves $P(1)$ by **evaluating LHS and RHS separately** ✓ assumes $P(k)$ is true ✓ writes LHS of $P(k+1)$ using RHS of $P(k)$ ✓ simplifies expression algebraically to one fraction ✓ factorises numerator with a factor of $(3k+2)$ ✓ obtains expression for RHS of $P(k+1)$ **written in terms of $k+1$** ✓ writes conclusion for whole proof

Question 6 (2.3.4, 2.3.6) **(7 marks)**

Use the principle of mathematical induction to prove the following statement:

$3^{2n+4} - 3^{2n}$ is divisible by 5 for all positive integers n.

Solution
Let $P(n)$ denote the proposition '$3^{2n+4} - 3^{2n}$ is divisible by 5' for all positive integers n. For $P(1)$: $3^6 - 3^2 = 720$ 720 is divisible by 5, so $P(1)$ is true. Assume that $P(k)$ is true for some positive integer k. Then $3^{2k+4} - 3^{2k} = 5m$, where $m \in Z$ Now For $P(k+1)$: $3^{2(k+1)+4} - 3^{2(k+1)}$ $= 3^{2k+6} - 3^{2k+2}$ $= 3^{2k+4} \times 3^2 - 3^{2k} \times 3^2$ $= 3^2(3^{2k+4} - 3^{2k})$ $= 9(5m)$ $= 5(9m)$ which is divisible by 5 $\therefore P(k+1)$ is true. Hence, by the principle of mathematical induction, $P(n)$ is true for all positive integers n.
Specific behaviours
✓ proves $P(1)$ by **evaluating** $3^6 - 3^2$ ✓ assumes $P(k)$ is true ✓ writes $P(k+1)$ **in terms of $k+1$** ✓ simplifies expression algebraically using index law for multiplication ✓ factorises expression with a factor of $(3^{2k+4} - 3^{2k})$ ✓ obtains expression for $P(k+1)$ **written as a multiple of 5** ✓ writes conclusion for whole proof

Additional working space

Question number: ________